1 Weierstrass Approximation Theorem

Theorem 2.5 in Text asserts uniform convergence of the Fourier series of a continuous, piecewise smooth, 2π-periodic function. As an application, we now prove a theorem of Weierstrass concerning the approximation of continuous functions by polynomials. It will be accomplished in three steps: First we approximate the given function by a continuous, piecewise linear function, then extend it to be an even function and finally apply Theorem 2.5.

Proposition 3.1. Let f be a continuous function on $[0, \pi]$. For every $\varepsilon > 0$, there exists a continuous, piecewise linear function g such that $|f(x) - g(x)| < \varepsilon/2$, $\forall x \in [0, \pi]$. Moreover, $g(0) = f(0)$ and $g(\pi) = f(\pi)$.

Proof. As f is continuous on $[0, \pi]$, it is also uniformly continuous on $[0, \pi]$. For every $\varepsilon > 0$, there exists some δ such that $|f(x) - f(y)| < \varepsilon/4$ for $x, y \in [0, \pi], |x - y| < \delta$. We partition $[0, \pi]$ into subintervals $I_j = [a_j, a_{j+1}]$ whose length is less than δ and define g to be the piecewise linear function satisfying $g(a_j) = f(a_j)$ for all j. For $x \in [a_j, a_{j+1}]$, g is given by

$$g(x) = \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j) + f(a_j).$$

For $x \in [a_j, a_{j+1}]$,

$$\begin{aligned} |f(x) - g(x)| &= \left| f(x) - \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j) - f(a_j) \right| \\ &\leq |f(x) - f(a_j)| + \left| \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j) \right| \\ &\leq |f(x) - f(a_j)| + |f(a_{j+1}) - f(a_j)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}, \end{aligned}$$

and the result follows.

Next we study how to approximate a continuous function by finite trigonometric series.

Proposition 3.2. Let f be a continuous function on $[0, \pi]$. For $\varepsilon > 0$, there exists a finite trigonometric series h such that $|f(x) - h(x)| < \varepsilon, \quad \forall x \in [0, \pi]$.

Proof. First we extend f to $[-\pi, \pi]$ by setting $f(x) = f(-x)$ (using the same notation) to obtain a continuous function on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$. By the previous proposition, we can find a continuous, piecewise linear function g such that $|f(x) - g(x)| < \varepsilon/2$ for all x. Since $g(-\pi) = f(-\pi) = f(\pi) = g(\pi)$, g can be extended as an even, continuous, piecewise smooth, 2π-periodic function. (A piecewise linear function is clearly piecewise smooth.) By Theorem 2.5 in Text, there exists some N such that $|g - S_N g(x)| < \varepsilon/2$ for all x. Therefore, $|f(x) - S_N g(x)| \leq |f(x) - g(x)| + |g(x) - S_N g(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. The proposition follows after noting that every finite Fourier series is a finite trigonometric series.

Theorem 3.3. (Weierstrass Approximation Theorem) Let $f \in C[a, b]$. Given $\varepsilon > 0$, there exists a polynomial p such that

$$|f(x) - p(x)| < \varepsilon \ , \quad \forall x \in [a, b].$$

Proof. Consider $[a, b] = [0, \pi]$ first. Extend f to $[-\pi, \pi]$ by reflection as before and, for $\varepsilon/2 > 0$, fix a finite trigonometric series h such that $|f(x) - h(x)| < \varepsilon/2$. This is possible due to the previous proposition. Here h is a finite cosine series $a_0/2 + \sum_{n=1}^{N} a_n \cos nx$. Using the fact that

$$\cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!},$$

where the convergence is uniform on $[-\pi, \pi]$, each $\cos nx, n = 1, \cdots, N$, can be approximated by polynomials. Putting all these polynomials together we obtain a polynomial $p(x)$ satisfying $|h(x) - p(x)| < \varepsilon/2$. It follows that $|f(x) - p(x)| \le |f(x) - h(x)| + |h(x) - p(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
When f is continuous on $[a, b]$, the function $\varphi(t) = f(\frac{b-a}{\pi}t + a)$ is continuous on $[0, \pi]$. From the last paragraph, we can find a polynomial $p(t)$ such that $|\varphi(t) - p(t)| < \varepsilon$ on $[0, \pi]$. But then the polynomial $q(x) = p(\frac{\pi}{b-a}(x - a))$ satisfies $|f(x) - q(x)| = |\varphi(t) - p(t)| < \varepsilon$ on $[a, b]$.

Note. Weierstrass Approximation Theorem is the first result concerning how to approximate functions by simpler ones. There is a branch of mathematics called Approximation Theory. Direct improvements on this theorem include the Bernstein's theorem and Jackson's theorem. Google for details if you are interested.

2 Weyl's Equivdistribution Theorem

In the proof of this theorem, Weierstrass Approximation Theorem is used in one step. See chapter 4 in Stein-Shakarchi.

3 Cesàro Mean and Fejér's Theorem

Theorem 2.5 concerning the uniform convergence of Fourier series requires the function under examination to be continuous, 2π-periodic and piecewise smooth. It was a main issue to determine whether uniform convergence still holds without the piecewise smooth condition. Eventually people constructed continuous, 2π-periodic functions whose Fourier series diverge at some point, showing that the piecewise smooth condition cannot be removed completely. One is referred to chapter 3 of Stein-Shakarchi and Part I, section 18, of Körner on these examples. On the other hand, as recent as 1965, L. Carleson proved a major result which implies that the Fourier series of any continuous, 2π-periodic function converges to itself "almost everywhere".

Going in another direction, one could relax the uniform/pointwise convergence of Fourier series by convergence in mean. Then a theorem of Fejér establishes mean convergence for every continuous, 2π-periodic functions.

Given an infinite series $\sum_{n=0}^{\infty} a_n$, we denote its N-th partial sum to be $s_N = \sum_{n=0}^{N} a_n$. Its N-th Cesàro sum is given by

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1} \ .$$

It is an exercise to show that s_N converges implies σ_N also converges (to the same limit), but the converse is not true. For instance, taking $\{a_n\} = \{-1, 1, -1, 1, \cdots\}$, $s_N = \{-1, 0, -1, 0, -1, 0, \cdots\}$ diverges but $\sigma_N = \{-1, -1/2, -2/3, -2/4, -3/5, -3/6, \cdots,\}$ converges to $-1/2$. Therefore, convergence in Cesàro sum is weaker than the usual convergence.

Theorem 3.4 (Fejér's Theorem). For every continuous, 2π-periodic function, the Cesàro sums of its Fourier series converges to it at every point.

Let $S_N f(x)$ be the N-th partial sum of the Fourier series of f and $\sigma_N f(x)$ be its N-th Cesàro mean. We first obtain a formula for $\sigma_N f$. Recall that we have

$$S_N f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(N+1/2)y}{\sin y/2} f(x+y)\, dy \ .$$

Using the formula

$$\sum_{n=0}^{N} \sin(n+1/2)y = \sin\frac{y}{2} + \sin(1+1/2)y + \cdots + \sin(N+1/2)y = \frac{\sin^2(N+1)y/2}{\sin^2 y/2} \ ,$$

(see exercise)

$$\sigma_N f(x) = \frac{1}{2\pi(N+1)} \int_{-\pi}^{\pi} \frac{\sin^2(N+1)y/2}{\sin^2 y/2} f(x+y)\, dy \ .$$

When $f(x) \equiv 1$, we know that $S_n 1(x) \equiv 1$, hence $\sigma_N 1(x) \equiv 1$ too. Using this we have

$$\sigma_N f(x) - f(x) = \int_{-\pi}^{\pi} F_N(y) \left(f(x+y) - f(x) \right) \, dy \ , \tag{1}$$

where the Fejér' kernel is given by

$$F_N(z) = \begin{cases} \dfrac{\sin^2 \left(N + \frac{1}{2}\right) z}{2\pi(N+1)\sin^2 \frac{1}{2} z}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

It is an even, continuous, 2π-periodic function. Unlike the Dirichlet's kernel D_N (see Notes 1), Fejér's kernel is non-negative. On the other hand, we also have

$$\int_{-\pi}^{\pi} F_N(y)\, dy = 1 \ , \quad \forall N \geq 1 \ .$$

Now we prove Fejér's Theorem. As f is continuous on $[-\pi, \pi]$, it is uniformly continuous on $[-\pi, \pi]$. Given $\varepsilon > 0$, we can fix some δ such that $|f(x+y) - f(x)| < \varepsilon/2$ for $y, |y| < \delta$ and $x \in [-\pi, \pi]$. We estimate the right hand side of (1) by splitting the integral into over $[-\delta, \delta]$ and over its outside. For the former we have

$$\begin{aligned} \left| \int_{-\delta}^{\delta} F_N(y)(f(x+y) - f(x))\, dy \right| &\leq \frac{\varepsilon}{2} \int_{-\delta}^{\delta} F_N(y)\, dy \\ &< \frac{\varepsilon}{2} \int_{-\pi}^{\pi} F_N(y)\, dy \\ &= \frac{\varepsilon}{2} \ . \end{aligned} \tag{2}$$

On the other hand, since $\sin y/2$ is bounded from below by a positive number for $y \in [-\pi, -\delta] \cup [\delta, \pi]$, the function $\sin^2(N+1)/2y/\sin^2 y/2$ is bounded by some number K. Letting $I = [-\pi, -\delta] \cup [\delta, \pi]$ and $M = \sup |f|$,

$$\begin{aligned} \left| \int_I F_N(y)(f(x+y) - f(x))\, dy \right| &\leq \frac{1}{2\pi(N+1)} \int_I K|f(x+y) - f(x)|\, dy \\ &\leq \frac{K \times 2M \times 2\pi}{2\pi(N+1)} \\ &= \frac{2KM}{N+1} \ . \end{aligned} \tag{3}$$

Putting (2) and (3) together,

$$\left| \int_{-\pi}^{\pi} F_N(y)(f(x+y) - f(x))\, dy \right| < \frac{\varepsilon}{2} + \frac{2KM}{N+1} \ .$$

As the second term on the right tends to 0 as $N \to \infty$, we conclude that $\sigma_N f$ converges to f uniformly as N goes to ∞. The proof of Fejér's Theorem is completed.